- There are 4 hours available for the problems.
- Every problem is worth 10 points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use a different sheet for each exercise.
- Clearly write DRAFT on any draft page you hand in.

MOAWOA

20 March 2015

Problem 1. Let $n \geq 4$ be an integer. Suppose that in a group of $2n$ people everyone speaks at least one of ℓ languages. Suppose that each of the ℓ languages is spoken by at least k people. We want these people to stand in a circle in such a way that each two neighbors have a common language.

(a) If $\ell = 2$, what is the minimal value of k such that this is always possible?

(b) If $\ell = 3$, what is the minimal value of k such that this is always possible?

Problem 2. Let $n > 1$ be an integer. Show that there exist positive integers a, b, c satisfying $a + b = n$ and $|ab - c^2| \leq 4$.

Problem 3. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ be the set of positive integers and let $f : \mathbb{N} \to \mathbb{N}$ be a bijective function.

(a) Is it possible that $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$ diverges?

(b) Is it possible that $\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$ converges?

Problem 4. Let G be a finite group with identity e and let H and K be subgroups of G such that $|H| \cdot |K| = |G|$ and $H \cap K = \{e\}$. Prove that $H' \cap K' = \{e\}$ for all conjugate subgroups H' and K' of H and K, respectively.
For a subgroup Y of a group X, a conjugate subgroup of Y is a subgroup of X that is of the form xYx^{-1} for some $x \in X$.

Problem 5. Let $n \geq 2$ be an integer and let $A = (a_{i,j})$ be a real $n \times n$ matrix with entries $a_{i,j}$ different from 0 that satisfy

$$a_{i,j}a_{i+1,j+1} - a_{i+1,j}a_{i,j+1} = ij$$

for all $i, j \in \{1, 2, \dots, n-1\}$. Determine the rank of A.

Problem 6. A biologist studies an exceptional bacterial species. When a bacterium of this species takes d minutes to divide, its two descendants take d and $d+1$ minutes to divide. The biologist starts with a single bacterium that takes 1 minute to divide.
Show that when the total number of bacteria becomes even for the n-th time, it stays even for exactly n minutes.