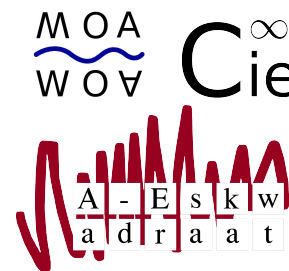


- There are 4 hours available for the problems.
- Every problem is worth 10 points.
- Be clear when using a theorem. When you are using an obscure theorem, cite a source.
- Use a different sheet for each exercise.
- Clearly write DRAFT on any draft page you hand in.



## MOAWOA

*20 March 2015*

**Problem 1.** Let  $n \geq 4$  be an integer. Suppose that in a group of  $2n$  people everyone speaks at least one of  $\ell$  languages. Suppose that each of the  $\ell$  languages is spoken by at least  $k$  people. We want these people to stand in a circle in such a way that each two neighbors have a common language.

- If  $\ell = 2$ , what is the minimal value of  $k$  such that this is always possible?
- If  $\ell = 3$ , what is the minimal value of  $k$  such that this is always possible?

**Problem 2.** Let  $n > 1$  be an integer. Show that there exist positive integers  $a, b, c$  satisfying  $a + b = n$  and  $|ab - c^2| \leq 4$ .

**Problem 3.** Let  $\mathbb{N} = \{1, 2, 3, \dots\}$  be the set of positive integers and let  $f : \mathbb{N} \rightarrow \mathbb{N}$  be a bijective function.

- Is it possible that  $\sum_{n=1}^{\infty} \frac{1}{nf(n)}$  diverges?
- Is it possible that  $\sum_{n=1}^{\infty} \frac{1}{n+f(n)}$  converges?

**Problem 4.** Let  $G$  be a finite group with identity  $e$  and let  $H$  and  $K$  be subgroups of  $G$  such that  $|H| \cdot |K| = |G|$  and  $H \cap K = \{e\}$ . Prove that  $H' \cap K' = \{e\}$  for all conjugate subgroups  $H'$  and  $K'$  of  $H$  and  $K$ , respectively.

*For a subgroup  $Y$  of a group  $X$ , a conjugate subgroup of  $Y$  is a subgroup of  $X$  that is of the form  $xYx^{-1}$  for some  $x \in X$ .*

**Problem 5.** Let  $n \geq 2$  be an integer and let  $A = (a_{i,j})$  be a real  $n \times n$  matrix with entries  $a_{i,j}$  different from 0 that satisfy

$$a_{i,j}a_{i+1,j+1} - a_{i+1,j}a_{i,j+1} = ij$$

for all  $i, j \in \{1, 2, \dots, n-1\}$ . Determine the rank of  $A$ .

**Problem 6.** A biologist studies an exceptional bacterial species. When a bacterium of this species takes  $d$  minutes to divide, its two descendants take  $d$  and  $d+1$  minutes to divide. The biologist starts with a single bacterium that takes 1 minute to divide.

Show that when the total number of bacteria becomes even for the  $n$ -th time, it stays even for exactly  $n$  minutes.